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RELIABILITY AND STRUCTURAL FATIGUE IN ONE-CRACK MODELS,(U)
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STRUCTURES REPORT 369

RELIABILITY AND STRUCTURAL FATIGUE IN ONE-CRACK MODELS

by

D. G. FORD

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RELIABILITY AND STRUCTURAL FATIGUE IN ONE-CRACK MODELS

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10 D. G. FORD

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SUMMARY

To be realistic, any model of structural fatigue should allow for attrition due to collapse and/or war damage, hijacking, etc. In one-crack structures, attrition adds a single term to the Fokker-Planck equation which statistically describes crack growth. This remarkably simplifies the reliability computation, and elucidates the interaction of fatigue, inspection, ultimate loads and other risks.

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NOTATION

a	Crack size
a_o	Crack size at initiation time
c	Critical crack size, $R(c)$ infinite
$E.$	Expectation
$f(a\|t)$	Conditional probability density of crack length (see $\phi(.)$, $\Phi(.)$)
$F(a) = \int_0^a f(a')\,da'$	Probability distribution of crack length
$f(t), f(t_o)$	Density of initial life t or t_o
$F(t)$	Distribution of initial life t
$H(t)$	Attrition or hijack factor of initial life survivorship function
$M.(u)$	Generic form of moment generating function, $E.\exp(u.)$
$R(a)$	Crack growth rate (local average)
$r(a)$	Risk function
$r_o = r(0)$	Risk in the absence of a crack "hijack risk"
t	Equivalent time or cycles
t_o, t	Initiation time
$t_a = t - t_o$	Elapsed time since initiation
T_a	Dominating time used in analysis of runaway cracks
$\phi(a)$, $\Phi(a)$	Marginal density and distribution of crack length at failure
$\phi(a\|t)$	Conditional density of crack length at a given life
μ_t	Arithmetic mean life to initial failure
$\cap$	Symbol for set intersection, used here as "and"

1. INTRODUCTION

Since the early 1960s research in fatigue prediction has divided into the distinct areas of crack growth, reliability or inspection and finally structural fatigue.[1,2,3] Until now, most of the models assumed for reliability have either been forced to make many simplifications[4] which have been criticised[5,6] or else the calculations have been inordinately lengthy and possibly ill-conditioned. These remarks apply more particularly to the assumptions used when inspections are included. Apart from this, many models involve structures or data that are highly idealised.[7,8,9]

Structural fatigue is now seen to involve the statistical interaction between initiation (damage) and crack growth at several fatigue critical regions in a structure. This concept developed naturally from the separation of fatigue into distinct phases of damage, cracking and final failure which is now the subject of reliability theory. The basic problem is now to find the multivariate distribution of crack lengths at any time. Although the general theory is multivariate, the single crack case is not trivial and forms useful illustrations.

In the earlier thinking of Ford,[1] the crack length distribution was used to find risk rates for reliability estimates in a separate but simultaneous computation. Thus until now general theory of structural fatigue has made no allowance for failures or retirements after inspection. In this paper, such an allowance is made for single cracks. In terms of structural fatigue theory the extension is almost trivial but the reliability theory is profoundly simplified and generalised. The success of this one-crack model means that prediction of structural fatigue must now intimately involve reliability or attrition, so that a multi-crack reliability theory is now needed.[8]

2. GENESIS OF NEW ONE-CRACK MODEL

The long fatigue life normally expected lends itself to measurement in terms of pseudo-times such as flights, service hours, years, etc. These are pseudo-times because they indicate length of service rather than physical time. Since practical time units contain many fatigue cycles it is mathematically convenient to let time be continuous, the common[4,5,6] but implied practice which we also follow. With such an idealisation, some events may be slightly anomolous in terms of discrete cycles.

We make a distinction between retirements after inspection at fixed times (termed "inspection" for brevity) and "attrition" at arbitrary times from a continuous combined risk of overload failure, war damage, hijacking, etc.

Following previous definitions,[1,2,3] fatigue is now assumed to proceed in three distinct stages. The first of these ends with the initiation (at length a_o) of a crack whose subsequent length a is assumed herein to be a deterministic function of elapsed time in the second stage. The third stage is attrition (including overload failure), and the structural fatigue problem is to find the corresponding probability density function $\phi(t)$ of life, which may allow for inspection. Because initiation is determined by cumulative damage, the pre-crack stage may also be termed the *damage* stage; "damage" does not refer to cracks, except that the next stage begins with a crack (of length a_o).

These stages are described by the three given functions $f(t)$, $R(a)$ and $r(a)$ relating to initiation, crack rate and risk. In practice of course, they are the output of preliminary estimations using relevant data and any fatigue method which averages the effects of load sequence. If inspections are included, an operating characteristic $P(a)$ is also required relating success of inspection to defect size (see page 9).

Thus, the model developed below produces a life distribution $\phi(t)$ from the data functions $f(t)$, $R(a)$, $r(a)$ and possibly $P(a)$. It should also be noted that, like t, the crack length a may be arbitrarily defined as long as it adequately describes the cracked structure. For example, crack length *or* area may be used, but the equally valid log (area) may be more convenient.

2.1 Inspections

If there is only one crack the damage is unaffected by crack growth for a given structure so that initial failure can be immediately predicted from an appropriate damage rule or, if inspections are of main interest, experimentally observed intiations may be used. If there is no inspection and no final failure then, with deterministic cracking, any crack that begins between any two initiation times retains the same probability measure from those times onwards (a runaway crack is a special case of this).

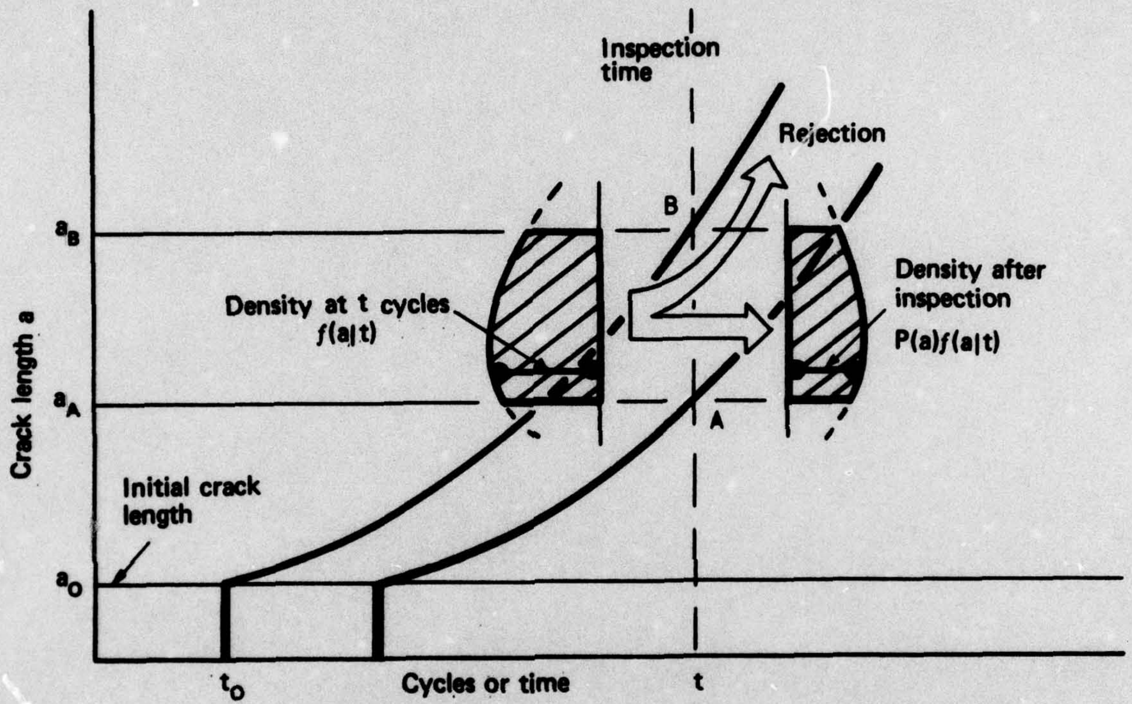


FIG. 1 EFFECT OF INSPECTION ON CRACK LENGTH DENSITY

In Figure 1 however, if there is an inspection at t the probability of having the crack between A and B will be discontinuously reduced at that time by an amount that depends on a_A, a_B the possible lengths at that time. The overall rejection rate is an average of such quantities.

Although deterministic cracking is assumed, local randomness can be included in a quasi-deterministic way and this has been done in the extensions. In the next section the possible cracks are also autonomous, i.e.

$$da/dn = R(a)$$

regardless of the time. If this is integrated then the inversion of

$$\int_{a_0}^{a} \frac{da}{R(a)} = t - t_0 = t_a \qquad \text{say}$$

indicates that any crack length in this one dimensional case depends only on the growth time t_a. Figure 2 shows some typical illustrations of this. However, although this is important in general structural fatigue it is not necessary here. The more general case is treated as an extension so that the first presentation is not too cluttered.

2.2 Attrition

If inspections are included in this manner, it is only fair that the continuous unscheduled "inspection" by the environment is also included. That is, the probability of overloads leading to failure should be covered; this is the subject of conventional reliability. As used below it will include the continuous risk of removal from any cause of interest, and the risk on an uncracked structure r_o will assume a particular importance. The latter will affect the probability density of initiation so that thus must also be included in the model.

In order to simplify the language it is convenient to introduce the term *attrition*, or (generalised) failure, for the process described by reliability theory. Thus

Risk function = Conditional probability of attrition.

2.3 Notational Conventions

In the following a generic notation will be used so that $f(.)$ and $F(.)$ will refer to the probability density and distribution functions of the arguments and the form of the function will differ with the argument. As a special case $\phi(.)$ refers to any density affected by attrition, including life distribution. $M.(u)$ will refer to the moment generating function of the random variable indicated by this subscript.

For example,

$$\begin{aligned} M_\phi(u) &= Ee^{u\phi} \\ &= \int_0^1 e^{u\phi}\, dF(\phi), \qquad (0 \leq F(\phi) \leq 1). \end{aligned}$$

In this standard definition we have also used a Stieltjes integral, mainly to abbreviate the formula. The reader who is not familiar with these will have no trouble if he remembers that when F is differentiable,

$$dF(\phi) = \frac{dF}{d\phi}\, d\phi;$$

otherwise $dF(\phi')$ is the concentrated probability $Pr(\phi = \phi')$. For similar brevity, Stieltjes integrals will be used below, when this is convenient, to improve the display of the relations obtained.

The main technical advantage of Stieltjes integrals is the simplification of expectations involving continuous and concentrated probabilities. When the latter are not zero, the continuous density will be termed *improper*, retaining the same notation. Thus in general

$$\int dF = 1 \qquad \text{but} \qquad \int_0^\infty f(\phi)\, d\phi \leq 1$$

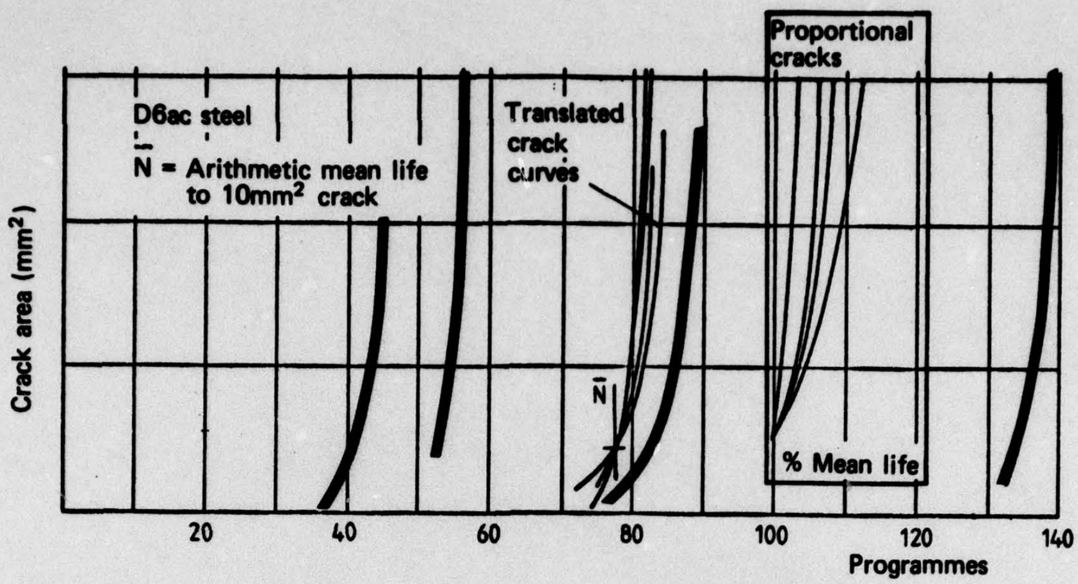
As a practical example we have for crack lengths

$$1 - \int_0^\infty dF(a|t) = \text{Probability that a structure will not survive to time } t \text{ (for a crack to start).}$$

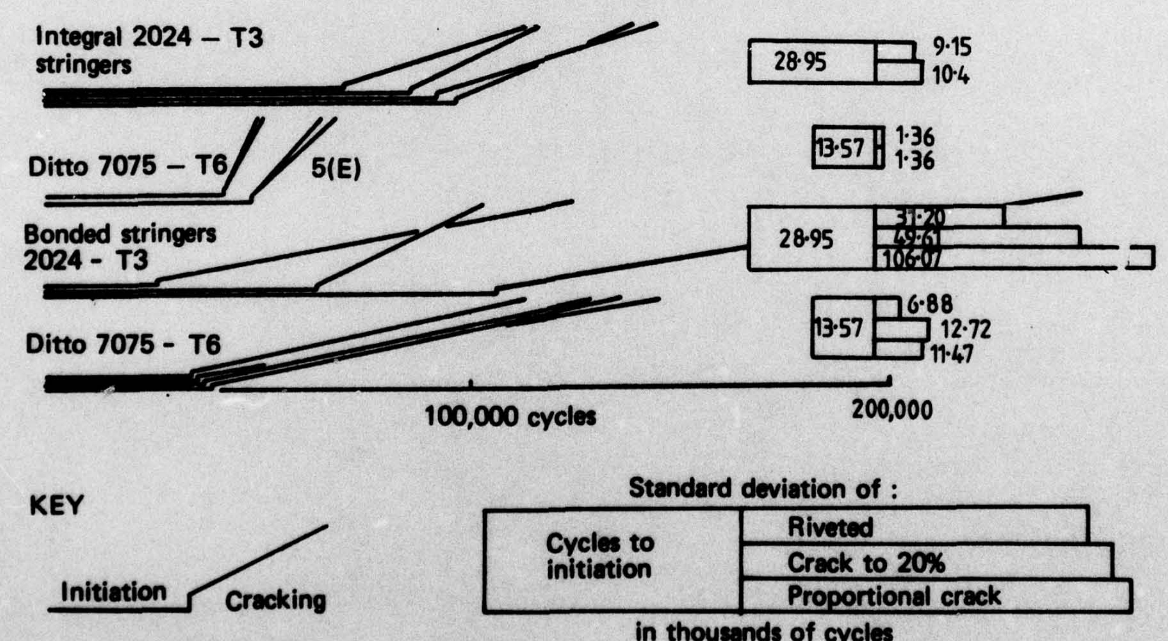
The vertical stroke above is the normal convention for conditional probability so that $a|t$ reads as "a given t". Without attrition, the conditioning event, that the structure has reached a life t, is a certainty. In our context, however, the conditioning event is this *or* prior attrition; $f(a|t)$ is then improper, summing only to the reduced probability of reaching life t.

3. COMBINED CRACKING AND ATTRITION

Consider the transition of possible cracks in the upper part of Figure 3 over a time dt.



(a) Channel specimens (ARL)



(b) Box beam results (NACA TN'S 3856, 4246)

FIG. 2 RELATIVE SCATTER OF INITIATION AND CRACKING TIMES

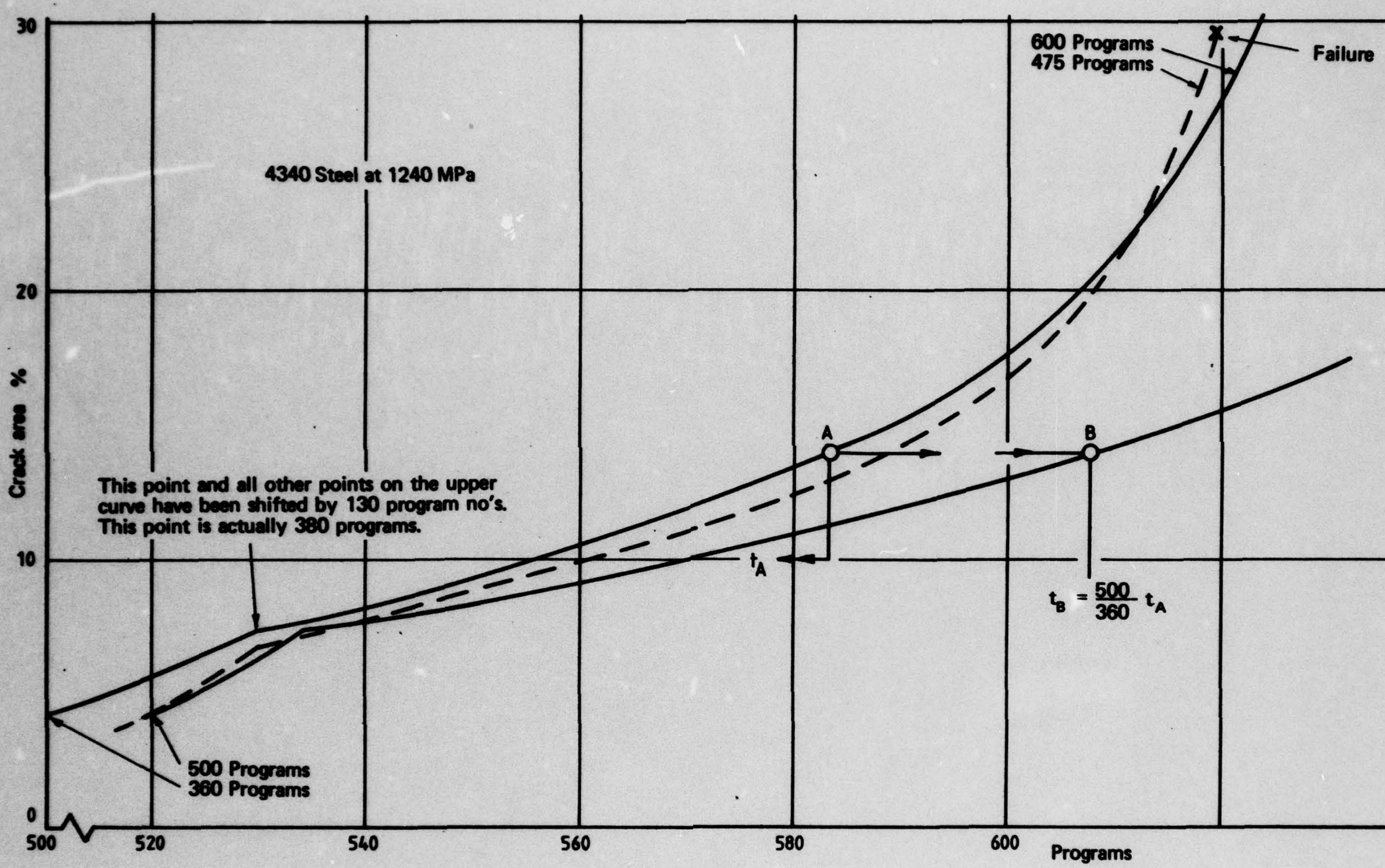


FIG. 2 (cont.) CRACKING IN TWO IDENTICAL SPAR BOOMS

If there is no attrition and f or $f(a|t)$ is the density of crack length then by an analogy with fluid flow[3] the constant probability that cracks will lie between the two shown may be expressed in the continuity equation

$$\frac{Df}{Dt} = -f \operatorname{div} R(a) \tag{3.1}$$

where Df/Dt is a total derivative along the crack trajectory and for single cracks $\operatorname{div} R(a) = dR/da$, a known function of a.

Attrition for this particular case will correspond to the risk rate $r(a)$ and the events of attrition and cracking are mutually exclusive. If the density f is allowed to be non-proper then attrition will decrease it at the rate $f(a|t)r(a)$ so that (3.1) with attrition becomes

$$\boxed{\frac{Df}{Dt} = -f\left(\frac{dR}{da} + r(a)\right)} \tag{3.2}$$

with

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + R(a)\frac{\partial f}{\partial a}$$

It is worth noting that if $\operatorname{div} R$ is retained (3.2) holds for several cracks but $r(a)$ becomes harder to calculate. Another point is that $r(a)$ is the *total* rate of attrition including that from causes not related to the growth of fatigue cracks or from various modes of failure. Statistically, the attrition simply adds to the effect of the dilatation dR/da.

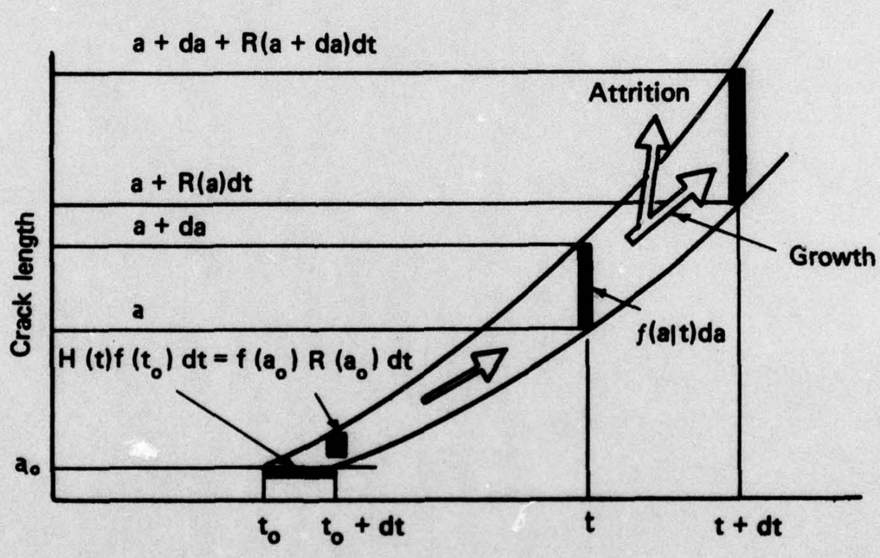


FIG. 3 DERIVATION OF DEGENERATE FOKKER-PLANCK EQUATIONS AND INITIAL CONDITIONS

3.1 Characteristic Equations

Equation (3.2) is a linear first order partial differential equation and is also the Fokker-Planck equation for the Markov process represented by the successive states of damage, crack length and total attrition as a function of crack length. (The usual second order term is absent because (3.2) is based on deterministic cracking without local randomness.)

From well known results[10,11,12] the characteristic equations are

$$dt = da/R(a) = -df/f[dR/da + r(a)]. \tag{3.3}$$

The first equality here indicates that the crack trajectories are base characteristics along which

$$-\frac{d}{da}\log f = \frac{dR/da + r(a)}{R(a)} \tag{3.4}$$

with the solution

$$\boxed{f = f(a_0|t_0)\frac{R(a_0)}{R(a)}\exp\left(-\int_{a_0}^{a}\frac{r(a)}{R(a)}da\right)} \tag{3.5}$$

where t_o is the time for initial failure or $a(t_o) = a_o$.

3.2 Initial Conditions

Equation (3.5) is the complete formal solution of (3.2) but the initial density $f_o = f(a_o|t_o)$ is still to be found. It follows from the same type of continuity considerations, but it must now be remembered that the density of initial failures is reduced in general by attrition of uncracked structures (e.g. static overload, hijacking, etc.). The attrition of cracked structures is already accounted for by (3.2) where $r(a)$ includes any risk of interest, *as well as* the failure of fatigue weakened structures.

Let $F(t)$ be the probability distribution of initial lives in the absence of attrition but in its presence suppose that the proportion of uncracked survivors is reduced to

$$H(t)(1 - F(t)). \tag{3.6}$$

Among these survivors there are two mutually exclusive forms of attrition, namely,

(*a*) Failure or removal of the uncracked structure with a risk rate $r_o = r(0)$, and

(*b*) Transition to being a cracked structure for which the risk rate is the same as it is without attrition, i.e. $f(t)/(1 - F(t))$.

For the survivors (3.6) the usual reliability equation then takes the form

$$-\frac{d}{dt}\log H(1-F) = r_0 + \frac{f(t)}{1 - F(t)} \tag{3.7}$$

in which only $H(t)$ is not known. If $H(0) = 1$ the solution is

$$H(t) = e^{-r_o t} \tag{3.8}$$

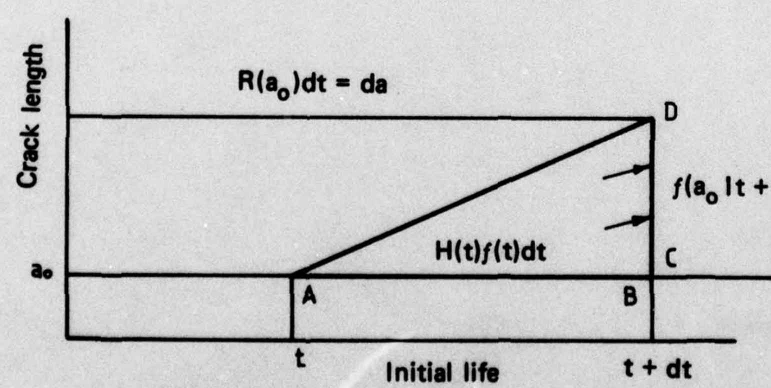


FIG. 4 BOUNDARY VALUE OF CRACK LENGTH DENSITY

It is now possible to find initial conditions for (3.5) from the preservation of probability between AB and CD in Figure 4. The probability of initiation between A and B is

$$Pr(\text{Survival}) \,.\, Pr(\text{Initiation}|\text{Survival})$$

$$H(t)(1 - F(t))\{1 - \exp[-f(t)dt/(1 - F(t))]\} = H(t)f(t)dt \tag{3.9}$$

to the first order. This is not obviously the same as the probability of a crack between C and D because some attrition may occur during the time dt with the risk $r(a_o+)$. If survival of this is included, like the risk in (3.9), and equated to the probability of an initial crack between C and D then the preservation of probability, with allowance for attrition, is expressible as

$$Hf\exp(-r(a_o+)dt)\,.\,dt = f(a_o|t + dt)R(a_o)dt.$$

The term for attrition here is of second order and the initial condition reduces to

$$f(a_o|t) = H(t)f(t)/R(a_o) \tag{3.10}$$

When this, with (3.8), is substituted into (3.5) we finally obtain

$$f(a|t) = \frac{e^{-r_o t_o} f(t_o)}{R(a)} \exp\left(-\int_{a_0}^{a} \frac{r(a)}{R(a)}\, da\right) \tag{3.11}$$

where the initial conditions are expressed on the axis of time or cycles and t_o is the initiation time corresponding to the crack $a|t$. The neglected second order term in (3.10) corresponds to simultaneous initiation and hijacking.

3.3 Distribution of Final Failures

First these are considered without inspections. For a given crack $a|t$ the risk of failure is $r(a)$ so that the probability of fatigue failure during $(t, t + dt)$ is $r(a)f(a|t)dt$, remembering that $f(a|t)$ here has been defined to allow for previous attritions. At time t these previous attritions (the chance that the structure will be overloaded, hijacked or removed *before* initial cracking), continue at the rate $r_o\exp(-r_o t)(1 - F(t))$. When this is added to the expected attrition of cracked structures the density of lives to failure from all causes appears as

$$\phi(t) = r_o e^{-r_o t}(1 - F(t)) + \int_{a_o}^{\infty} r(a)f(a|t)da \tag{3.12}$$

which includes infinite cracking rates. The notation ϕ is a convention to indicate the presence of attrition. The occurrence of infinite cracking rates at finite lengths is the case of runaway cracking and is discussed below. The two terms in (3.12) form improper densities for failure of uncracked and cracked structures respectively. Now consider the second term and substitute for the density from (3.11). This corresponds to the attrition of cracked structures so that

$$\phi(t \cap \text{Cracking}) = \int_{a_o}^{\infty} e^{-r_o t_o} f(t_o)\cdot \frac{r(a)}{R(a)} \exp\left(-\int_{a_0}^{a} \frac{r(c)}{R(c)}\, dc\right) da \tag{3.13}$$

and for autonomous cracks the integrals may be simplified by a change of variable. Replace c by t_a, the time that the crack requires to grow from a_o to c. This is defined for an autonomous crack and a moment's thought will convince the reader that $da/dt_a = R(a)$. With these changes, and the hijack term of (3.12), (3.13) becomes

$$\phi(t) = r_o e^{-r_o t}(1 - F(t)) + \int_{0}^{\infty} e^{-r_o(t - t_a)} f(t - t_a) \cdot r(t_a) \exp\left(-\int_{0}^{t_a} r(t')dt'\right) dt_a \tag{3.14}$$

In the integral the factors depending on $t - t_a$ form an improper density of initiation, since the first term allows for prior attrition by ultimate loads. The remaining factors could obviously be the result of attrition with the risk $r(t_a) \equiv r(a(t_a))$ on the set of structures which start to crack at $t_a = 0$; they are the probability density of the time spent with cracks in the structure before final failure. If $r_o = 0$ then the convolution integral in (3.14) would be a proper

density reflecting the fact that the total life is the sum of the initiation period and an independently distributed time of crack growth.

3.4 Moment Generating Function for Lives

With (3.14) the life distribution (3.12) becomes

$$\phi(t) = r_o e^{-r_o t}(1 - F(t)) + e^{-r_o t_o} f(t_o) * f(t_a) \tag{3.15}$$

so that the Moment Generating Function Eexp(ut) is

$$M_\phi(u) = r_o \int e^{(u-r_o)t}(1 - F(t))dt + \int dt \int_0^1 e^{ut} e^{-r_o(t-t_a)} f(t - t_a)\, dF(t_a)$$

$$= \frac{r_o}{u - r_o} \int (1 - F(t))\, de^{(u-r_o)} + \int\int e^{(t-t_a)(u-r_o)}\, dF(t - t_a) e^{ut_a}\, dF(t_a)$$

$$\boxed{= M_t(u - r_o) M_a(u) + \frac{r_o}{u - r_o}(M_t(u - r_o) - 1)} \tag{3.16}$$

where M_t and M_a refer to the initiation and cracking times respectively. When there are no ultimate loads $r_o = 0$ and the independence of t and t_a is confirmed. The result is exact for autonomous cracks and some expansions of the cumulants of final life are shown in Table 1.

3.5 Allowance for Inspections

The effect of an inspection may be regarded as a concentrated attrition applied instantaneously. In terms of the Fokker-Planck Equation (3.2) and Figure 5 the boundary conditions are altered for different crack lengths along the inspection boundaries $t = T_i$. The computation of $f(a|t)$ and therefore of $\phi(t)$ proceeds in turn from regions $I_o, \ldots, I_i$ to I_{i+1} with changes of boundary conditions at T_i.

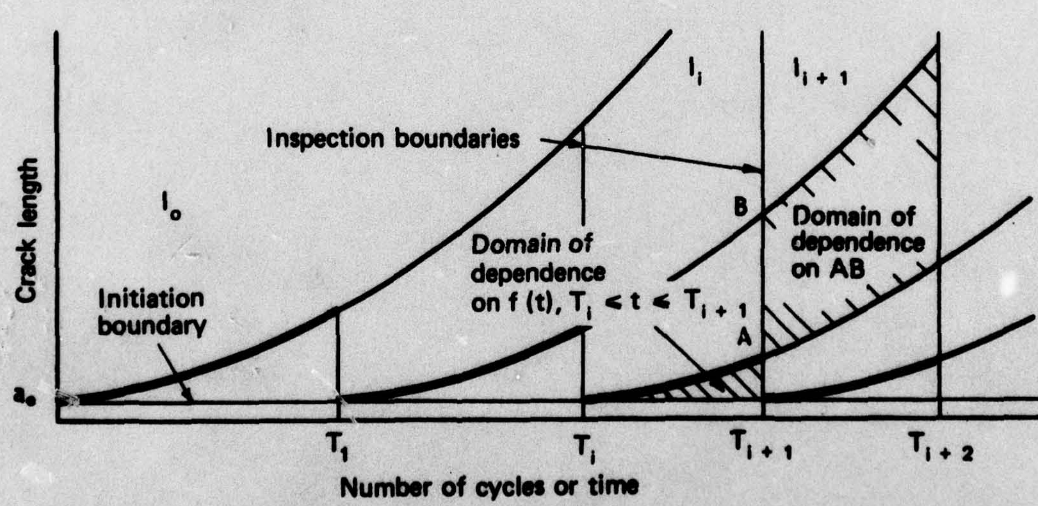


FIG. 5 DEPENDENCE ON INSPECTIONS

Let $P(a)$ be the operating characteristic of the method of inspection so that

$$P(a) = Pr\,(\text{Not acting about a crack}|\text{Length } a).$$

In general $P(a)$ will be the result of operating experience but the more exotic possibilities of

finding cracks and not acting or randomly rejecting structures can be included in subsidiary calculations for it. After the first inspection the boundary conditions along t_1 become

$$f(a|T_1+) = P(a)f(a|T_1-)$$

but this will only affect the cracks that have already begun, that is those in the domain of dependence

$$(a, t) \in I_1 \text{ and } a \leqslant a(t - T_1)$$

which is that part of I_1 above the characteristic crack trajectory starting at (a_o, T_1) (Fig. 5). Similarly the trajectories from the other (a_o, T_i);s will divide all the I_i into other domains of dependence which depend on conditions set at the inspection boundaries $t = T_i$, $a \leqslant a_o$ or else the initiation boundaries $T_i < t \leqslant T_{i+1}$, $a = a_o$. Along each of the latter the probability of initiation will be greatly affected by the inspection policy. The study of these policies and their effect is related to renewal theory.[13] Because little is known at present and the subject promises to be vast it will not be pursued here except for the mention of four typical possibilities. These are

(*a*) the structure is removed from service;

(*b*) it is repaired as new, i.e. the damage is the same as that of a new structure;

and finally there are the more difficult

(*c*) repaired to $a = 0$ but with residual fatigue damage (this means that the repair will tend to fail more rapidly than originally);

(*d*) replaced by new structure selected at random from the population.

4. EXTENSIONS

For the model above the obvious need is more investigation of inspection policies. However, there are other simple extensions which are either interesting or related to current practice.

The first of these is that the initial crack length a_o could be taken as zero in the context of the present development although there are physical reasons[1,2,3] why it should not be.

4.1 Runaway Cracks

In Forman, Kearney and Engle's relation[14] for da/dt and for practical purposes in some experimental results, it is possible for $R(a)$ to become infinite at some finite length c, after the growth time t_c say. If this happens (defining a "runaway crack") then obviously such structures survive no longer; in the basic statistical terms $r(c) = 1$ and it will transpire that $c \to \infty$ makes no difference. The effect of this is to telescope all of the failures which would have occurred later on (had da/dt not become infinite) into the instant at $t_o + t_c$.

This is the continuous idealisation of the final cycle in a discrete sequence of loads.

4.1.1 Discrete Analogy

It may be argued that a risk approaching $r(c)$ when $t < t_o + t_c$ would leave no survivors for such a wholesale demise when the risk become unity. However, we shall demonstrate the effect for physically more realistic discrete time, i.e. individual load cycles. It is sufficient to postulate a fixed crack always "beginning" at the origin with the induced risks $r_o, r_1 \ldots r_{N-1}$ and of course $r_N = 1$.

Then for $j \geqslant N - 1$

$$Pr(\text{Surviving } j \text{ cycles}) = (1 - r_o) \ldots (1 - r_j) = (1 - F_j), \text{ say} \tag{4.1}$$

In particular the nature of r_j ensures strictly that

$$0 < 1 - F_{N-1} < 1, \tag{4.2}$$

since N is finite. The attrition on the next cycle is complete,

$$r_N(1 - F_{N-1}) = 1 - F_{N-1},$$

and the range of the life distribution is $(0, N)$ with

$$f_N = 1 - F_{N-1}, \tag{4.3}$$

the telescoping effect.

If this simplified case is repeated in continuous time then (4.1) is replaced by

$$1 - F(t_c) = \exp\left(-\int_0^{t_c} r(t_a)dt_a\right) > 0 \tag{4.4}$$

as in (4.2) if the integral converges, as it may.

As in (4.3) $f(t_c)$ is a concentrated probability, an exceptional case in continuous time, but only because (4.4) is an approximation to (4.1). It is easily shown that there exists a believable $r(t)$ for which (4.4) uniformly exceeds (4.1).

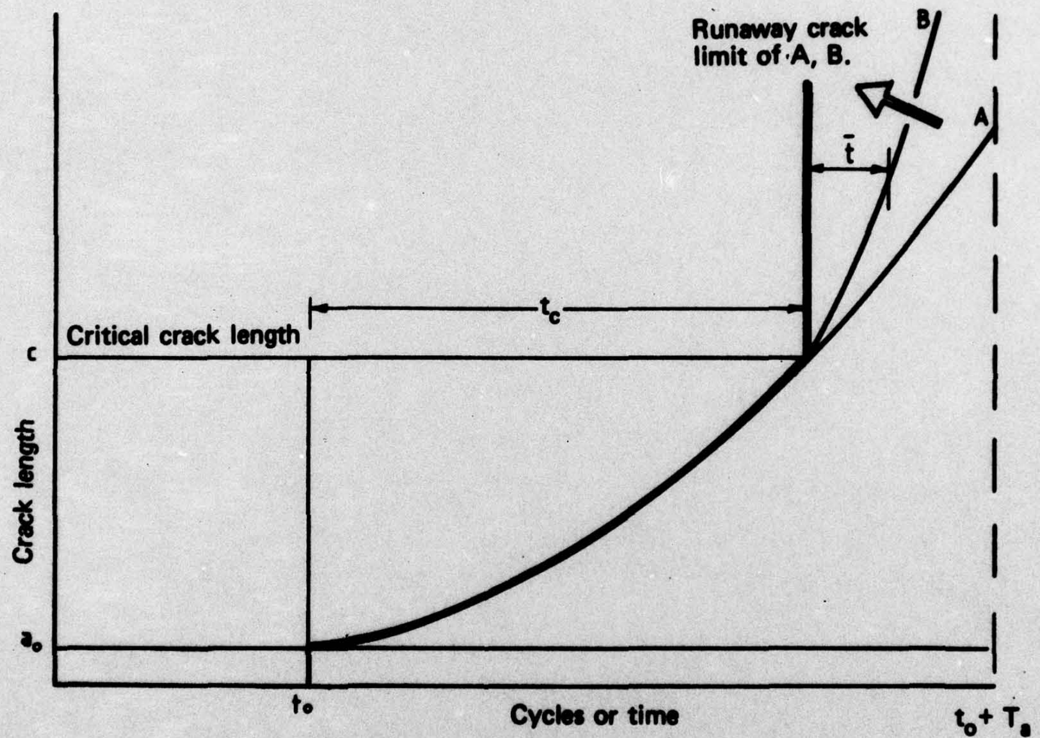


FIG. 6 RUNAWAY CRACKS

4.1.2 Runaways as a Limiting Crack Law

In this section, runaway cracks will be approached as the limiting case of an inequality for the distribution function of crack lengths $F(a|t)$. For this purpose suppose that $c = a(t_c)$ is some arbitrary length above which the rate is "fast" so that $t(\infty) - t(c)$ is "short".

In the notation of (3.15) let us divide the infinite range in (3.14) at the arbitrary times t_c and $T_a > t_c$. The first of these will become the time until a crack runs away whilst T_a is to allow for the infinite limit in (3.14).

Thus (3.14) may be written

$$\phi(t\cap C) = \int_0^{t_c} + \int_{t_c}^{T_a} + \int_{T_a}^{\infty} e^{-r_0(t-t_a)} f(t-t_a)\,dF(t_a)$$

$$= \int_0^{t_c} e^{-r_0(t-t_a)} f(t-t_a)\,dF(t_a) + e^{r_0\bar{t}-r_0(t-t_c)} f(t-t_c-\bar{t})\,[F(T_a)-F(t_c)]$$
$$- O[e^{-r_0T_a} f(0)(1-F(T_a))]$$

where $0 \geqslant \bar{t} \geqslant T - t_c$ by the mean value theorem. Because $f(t)$ is zero when $t < 0$ this inequality can be sharpened to $0 \leq \bar{t} < t - t_c$ since $Pr(\bar{t} + t_c > t)$ is then zero.

The third term is essentially positive and its neglect indicates that always

$$\phi(t\cap C) \leq \int_0^{t_c} \ldots .\, dF(t_a) + e^{M} \cdot e^{-r_0(t-t_c)} f(t-t_c)\,[F(T_a)-F(t_c)] \tag{4.5}$$

with equality when $F(T_a) = 1$. This happens when T_a is infinite or at finite values when $r(c) = 1$, i.e. there is runaway cracking. This is the case we discuss. In (4.5)

$$M = r_0\bar{t} + \ln\{f(t-t_c-\bar{t})/f(t-t_c)\}$$
$$= r_0\bar{t} + \ln\{1 - \bar{t}f'/f\} + O(\bar{t}^2) \sim \bar{t}(r_0 - f'/f(t-t_c)).$$

So far the introduction of $\bar{t}$ or M has avoided the use of unit risk. In this Section risk must be correlated with cracking so that $r(c) \to 1$ is implied by $\bar{t}, M \to 0$, i.e. certain failure follows instantaneous fracture.

For runaway cracks in continuous time then $\phi(t \cap C)$ becomes

$$\int_0^{t_c} e^{-r_0(t-t_a)} f(t-t_a)\,dF(t_a) + e^{-r_0(t-t_c)} f(t-t_c)(1-F(t_c)).$$

From Section 3.3

$$1 - F(t_c) \equiv \int_{t_c}^{\infty} dF(t_a) = \exp\left(-\int_0^{t_c} r(t_a)\,dt_a\right) \tag{4.6}$$

and in the limit (3.12) or (3.15) take the most general form

$$\phi(t) = r_0 e^{-r_0t(1-F(t))} + \int_0^{t_c} e^{-r_0(t-t_a)} f(t-t_a) \cdot r(t_a) \exp\left(-\int_0^{t_a} r(t')dt'\right) dt_a$$

$$+ e^{-r_0(t-t_c)} f(t-t_c) \cdot \exp\left(-\int_0^{t_c} r(t')dt'\right) \tag{4.7}$$

The last term is the "risk of fatigue fracture" as defined by Payne and Diamond.[4] The moment generating function (3.16) is not altered but if $R(a(t_c))$ is infinite then $M_a(u)$ needs to be computed using the concentrated probability (4.6) at t_c.

4.1.3 Discreteness Errors

In (4.5) $\exp M \sim 1 + M$ indicates the effect of fracture not being instantaneous. If $r(c) = 1$ then $\bar{t}$ cannot "exceed" one load cycle and with this extreme bound we now show that M is negligible.

Suppose that unit time allows an average of m cycles so that heuristically

$$r_0 = mr_m(0) \qquad\qquad f'/f = \Delta f_m/f_m$$

where the subscript m indicates discrete load quantities. Substitution with $\bar{t} \sim 1/m$ then indicates that $M < r_m(0) - \Delta f_m/mr_m(t-t_c)[1 - F(t-t_c)]$ which is small for practical cases.

4.2 Non-Autonomous Cracks

So far, the crack rate has depended only on the current length so that any possible growth curve is a translation in time or cycles of one that begins at the origin. In practice, the rate, say $da/dt = R(a, \omega)$, depends randomly on the particular specimen, here itemised as ω. For a two stage model of fatigue this variability can either be neglected in comparison to the scatter from the damage phase or else (4.7) can be repeated for different cases and averaged. Within the present model the crack rate can also be modified to allow approximately for random crack rates.

In other cases, notably for some welded structures, it may be more realistic to assume that a crack begins immediately or at some fixed time and another further idealisation would be that the time to a given crack length has some given probability density.[4,5]

Such cases are included by taking a crack rate $R(a, t)$ depending both on time and length which also generalises the two stage model. Equation (3.2) still holds together with the subsequent argument except that the dilatation of R is now a partial derivative depending on time. The first effect of this is that (3.5) fails, which in turn means that the simple form (3.16) for the moment generating function must be abandoned; the propagation time and initiation are *not* independent.

However, some salvage is possible. Crack growth trajectories are still characteristics and along these

$$-\frac{d}{dt}\log f = \frac{\partial R(a, t)/\partial a + r(a)}{R(a, t)} \tag{4.8}$$

leading to a formal solution with the previous boundary conditions. Consider the total derivative of R along the crack trajectory,

$$\frac{DR}{Da} = \frac{\partial R}{\partial a} + \frac{\partial R}{\partial t}\frac{dt}{da}.$$

Replacing $\partial R/\partial a$ in (4.8) and integrating

$$\log f = A - \log R - \int_{a_0}^{a} \frac{r(a)}{R(a, t)}\, da + \int_{a_0}^{a} R^{-2}\frac{\partial R}{\partial t}\, da.$$

Initially

$$\begin{aligned} f_0 &= e^{-r_0 t_0} f(t_0)/R(a_0, t_0) \\ &= A/R(a_0, t_0), \qquad t_0 \text{ depending on } a, \end{aligned}$$

whence

$$f(a|t) = e^{-r_0 t_0} f(t_0)\,(R(a, t))^{-1} \exp\left(-\int_{a_0}^{a} \frac{r(a)}{R(a, t)}\, da + \int_{a_0}^{a} R^{-2}\frac{\partial R}{\partial t}\, da\right) \tag{4.9}$$

which corresponds to (3.11). The integrals here are defined by relating a and t along the characteristics; (3.12) follows as before with the changes above.

The modifications needed to display runaway cracking may also be extended to non-autonomous cracks. The argument is similar but must begin at (3.12) which now becomes

$$\phi(t) = r_0 e^{-r_0 t}(1 - F(t)) + \left(\int_{a_0}^{c} + \int_{c}^{\infty}\right) e^{-r_0 t_0} f(t_0)\frac{r(a)}{R(a, t)} \times$$

$$\times \exp\left(-\int_{a_0}^{a} \frac{r(a)}{R(a, t)}\, da\right) \exp\left(\int_{a_0}^{a} R^{-2}\frac{\partial R}{\partial t}\, da\right)$$

Since $R(c)$ is infinite the last term becomes unity for the range of integration (c, ∞) and this leaves

$$\frac{r}{R} e^{-\int (r/R) da}$$

as a probability density from an attrition along the characteristic through (c, t) with the risk rate r/R. In addition there is no longer any reason why c need be independent of t. All of these considerations lead to

$$\phi(t) = r\, e^{-r_0 t_0}(1 - F(t)) + \int_{a_0}^{c} e^{-r_0 t_0(a)} f(t_0(a)) \exp \int_{a_0}^{a} \frac{\partial R}{\partial t} \frac{da}{R^2}\, dF$$
$$+ e^{-r_0 t_0(c)} f(t_0(c))(1 - F(a|a = c)), \qquad c = c(t). \tag{4.10}$$

as the most general one-crack form of (4.2), where

$$f(a) = 1 - \exp\left(-\int_{a_0}^{a} \frac{r(a')}{R(a', t')}\, da'\right) \tag{4.10A}$$

along the crack trajectory through (a, t).

4.3 Distribution of Failing Loads

In the context of this paper failing loads may be inferred from the crack length when the structure fails so that we address the problem of finding the density of crack lengths at failure. With respect to the overall probability

$$r(a) f(a|t) = Pr(\text{Attrition} \cap \text{Survival to } t \cap \text{Crack length } a).$$

By definition

$$\phi(t) = Pr(\text{Attrition at } t)$$

in overall measure whence the crack length density conditional upon attrition,

$$\phi(a|t) = r(a) f(a|t)/\phi(t) \tag{4.11}$$

at a given number of cycles; $f(a|t)$ refers to cracks which may be merely present at time t. For autonomous cracks and $a_0 \geqslant a \geqslant c$, substitution from (3.11) produces

$$\phi(t)\phi(a|t) = e^{-r_0 t_0} f(t_0) \frac{r(a)}{R(a)} \exp\left(-\int_{a_0}^{a} \frac{r(a)}{R(a)}\, da\right)$$
$$= e^{-r_0(t - t_a)} f(t - t_a) \frac{r(a)}{R(a)} \exp\left(-\int_{a_0}^{a} \frac{r(a)}{R(a)}\, da\right)$$
$$= e^{-r_0(t - t_a)} f(t - t_a) f(t_a)/R(a) \tag{4.12}$$

with the change of variable $a \to t_a$ used for (3.14).

In words, the bivariate density of life and crack length, for life at final failure, is the product of growth time density and the density of initiation of the characteristic divided by the current rate of cracking.

4.3.1 Zero Cracks

The density (4.12) should be augmented by concentrated probabilities that the cracks in the failed structure either did not exist or possibly were runaway cracks.

The first of these is the conditional probability $\Phi(0|t)$ of attrition by the risk r_0. From the first term of (3.15) or (4.10), normalised by $\phi(t)$,

$$\phi(t)\Phi(0|t) = r_0 e^{-r_0 t}(1 - F(t)) \tag{4.13}$$

which may be integrated as it stands for the marginal probability of failure or removal without cracking. Partial integration with respect to the factor in r_0 leads to

$$\boxed{\begin{aligned}\Phi(0) &= 1 - M_t(-r_o)\\ &= r_o\mu_t + O(r_o^2),\end{aligned}} \tag{4.14}$$

where μ_t = Mean cycles to initiation.

It is suggested that (4.14), the product of arithmetic mean life to initiation with risk rate of uncracked structures, is an appropriate measure of "fatigue sensitivity" since it is the probability that failure will not be related to fatigue.

4.3.2 Runaway Cracks

By arguments the same as above the last term of (4.7) may be normalised to the conditional probability of runaway cracking whence

$$\phi(t)(1 - \Phi(c|t)) = e^{-r(t-t_c)}f(t - t_c)(1 - F(t_a|a = c)) \tag{4.15}$$

with $f(t_a)$ from (4.6). Since $f(t)$ is zero for negative times this integrates over t to the marginal probability of runaway cracking

$$\boxed{\begin{aligned}(1 - \Phi(c)) &= M_t(-r_o)(1 - F(t_a|a = c))\\ &\approx (1 - r_o\mu_t)(1 - F(t_a|a = c)).\end{aligned}} \tag{4.16}$$

When $R(c)$ is not necessarily infinite but large enough to be considered dangerous (4.16) may be regarded as a rough measure of "fatigue danger". The interpretation is precise for runaway cracks but otherwise clouded by considerations such as those of Section 4.1.

4.3.3 Marginal Densities

When (4.12) is integrated with respect to $t - t_a$ one obtains the marginal density

$$\phi(a) = M_t(-r_o)f(t_a)/R(a) \tag{4.17}$$

which follows from (3.11) and (4.11). Inspection of these original equations indicates that, just as $f(t_a)$ is the outcome of an attrition process with the risk $r(a)$ and $a = a(t_a)$, $f(t_c)/R(a)$ is both the transformation of that density to that of $a(t_a)$ and the result of attrition along crack length with the risk $r(a)/R(a)$ (see Section 3.3).

It can be shown that (4.14), (4.16) and (4.17) provide the complete marginal distribution of length at failure.

5. DISCUSSION

The preceding evolution of two stage fatigue depends only on the density of initial life $f(t)$ and the average crack rate $R(a)$. If failure is included a risk $r(a)$, averaged over stress sequence, material toughness and untoward circumstance is also needed. These three functions then completely define fatigue and life of the structure. They may be estimated by any standard method, experimental or theoretical.

The computer programs may also allow for inspections if the operating characteristic $P(a)$ of the overall inspection method is known.

Most current experimentation about crack rates tacitly assumes that, apart from the stress, they depend only on the configuration of the crack at the time. This is the assumption of autonomous cracking which is suited to the use of fracture mechanics and supported by the small selection of data in Figure 2 and most of that in the literature.

For single cracks we have shown that the assumption of autonomous cracking, which is essentially that fatigue has two stages, leads to remarkable simplifications of the associated reliability theory provided that this is incorporated into the structural fatigue analysis. There is a parallel increase in understanding of the interaction of ultimate loads and the fatigue of structures which may be summarised by Equations (3.16) and (4.7).

Both of these are exact for a very general model with automatic allowance for the increasing strength of structures with time as the weaker ones are weeded out by higher risks.[6] Some infor-

mation about this will be contained in the crack length distribution results of (4.12), (4.13) and (4.15) though interpretation in terms of strength depends on the nature of the subsidiary problem of estimating the risk $r(a)$. In the same section it can be seen that the marginal distribution of strength particularly the virgin strength and that for runaway cracks provides natural measures of the sensitivity of the structure both to fatigue in general and to fatigue which may be regarded as dangerous.

Finally, many of the results can be extended to cracks that are not autonomous with no loss of accuracy and very little loss of simplicity.

The representation of the crack length density as a Markov process governed by a partial differential equation also allows an exact solution for the results of general inspections with realistic operating characteristics. Although this is exact in principle the calculations involve following boundary conditions across many contiguous domains of dependence. For single cracks the optimisation of such computations for either speed or accuracy is the major question in the application to structural fatigue and reliability.

For structures with several cracks the success of the results here demonstrates that any theory of structural fatigue must intimately include reliability and the latter subject must now become multi-dimensional. In practice many structures fail by a single dominant crack and, if this is so, many of the present results will apply if the initial crack length a_o is taken to be some length exceeding that of non-dominant cracks.

When they coalesce, non-dominant cracks will thus affect the stress distribution and qualify for our definition of a crack. Before this, however, their progress would be described as contributing to damage.

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TABLE 1

Cumulants of Fatigue Life in Terms of Initiation and Crack Time Moments

$$M_t(u) = \text{MGF of cycles to initiation}$$

$$M_t^{(k)}(-r_o) = \mu_k + O(r_o) \quad \text{where } \mu_k = k\text{th moment, } \mu_o \equiv 1.$$

$$M_a(u) = 1 + \alpha_1 u + \tfrac{1}{2}\alpha_2 u_2 \ldots, \qquad \alpha_o \equiv 1,$$

$$= \text{MGF of cracking time, moments } \alpha_k.$$

Put

$$m_k \equiv M_t^{(k)}(-r_o); \qquad A_k \equiv \alpha_k - k\alpha_{k-1}/r_o,$$

where

$$r_o \equiv \text{Risk of loads above ultimate.}$$

Then for total life:

$\kappa_1 = m_o A_1 + r_o^{-1}$ (Mean)

$\kappa_2 = 2m_1A_1 + m_oA_2 - m_o^2A_1^2 + r_o^{-2}$ (Variance)

$\kappa_3 = 3m_2A_1 + 3m_1A_2 + m_oA_3 - 6m_om_1A_1^2 - 3m_o^2A_1A_2 + 2m_o^3A_1^3 + 2/r_o^3$

$\kappa_4 = 4m_3A_1 + 6m_2A_2 + 4m_1A_3 + m_oA_4 - 12m_om_2A_1^2 - 24m_om_1A_1A_2 + 24m_o^2m_1A_1^3$

$- 12m_1^2A_1^2 - m_o^2\{4A_1A_3 + 3A_2^2\} + 12m_o^3A_1^2A_2 - 6m_o^4A_1^4 + 6/r_o^4$

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16. ABSTRACT

To be realistic, any model of structural fatigue should allow for attrition due to collapse and/or war damage, hijacking, etc. In one-crack structures, attrition adds a single term to the Fokker-Planck equation which statistically describes crack growth. This remarkably simplifies the reliability computation, and elucidates the interaction of fatigue, inspection, ultimate loads and other risks.

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